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Hierarchical lattices: some examples with a comparison of intrinsic dimension and connectivity and Ising model exponents

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Abstract. The recently recognised class of hierarchical lattices is examined through a number of examples. Definitions of length, intrinsic dimension and connectivity are made and used to discuss variations of exponents calculated on the Ising model. Generalisations of regular lattice results are found for exponents at discontinuity fixed points and in $1+\varepsilon$ dimensions.

Recently many hierarchical lattices which support exact solution have been introduced (Gefen *et al* 1980, Dhar 1977, Nelson and Fisher 1975, Berker and Ostlund 1979, Kaufman and Griffiths 1981, 1982a, b). Hierarchies are defined as the infinite limit of an iterative generation of larger and larger lattices. Those studied will be termed bond hierarchies, starting from a single bond at each iterative step a larger lattice is formed by decorating each bond of the previous lattice with some basic cell (see figure 1). Decimation transformations on hierarchies trivially factor and constitute an exact renormalisation group.

Hierarchies are often highly inhomogeneous and lack translational invariance. The lattices can support a wide variety of phase transitions although they can also exhibit unusual features (Kaufman and Griffiths 1981, 1982a). Many decimation approximations on regular lattices (Barber 1975, Migdal 1975, Kadanoff 1976, Reynolds *et al* 1977) consitute exact solutions on hierarchies.

In this work exploration is made of the relationships between critical exponents of the Ising model on the hierarchies and two parameters, the intrinsic dimension (McKay *et al* 1982) and the connectivity (Gefen *et al* 1980). There are many, as yet unclassified, varieties of hierarchies. The examples studied here were chosen for their intuitive simplicity and involve only nearest-neighbour couplings. Definitions below are restricted to these examples.

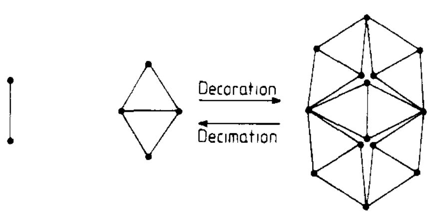

Figure 1. The first two steps in the iteration sequence of a hierarchy (cell 2a below).

A classification is introduced: set the cell dimension, d_c, as that of the basic cell. Some examples with $d_c = 1$ to 4 are shown in figures 2 and 3. Each cell has two special vertices, *the nodes*, between which the decoration is defined (cf 2a with figure 1). The parametrised family of cells in (1a) of figure 2 are members of the Migdal–Kadanoff hierarchies (MKH) on which the well known Migdal–Kadanoff approximations are exact (Berker and Ostlund 1979). Some of the examples are finite cluster approximations on regular lattices. Two parameters useful below are g, the aggregation number, the number of bonds on the basic cell, and q the minimum cut, the minimum number of bonds which need be cut on the basic cell to separate the nodes. On the MKH $g = MA$ and $q = M$; other values are given in table 1.

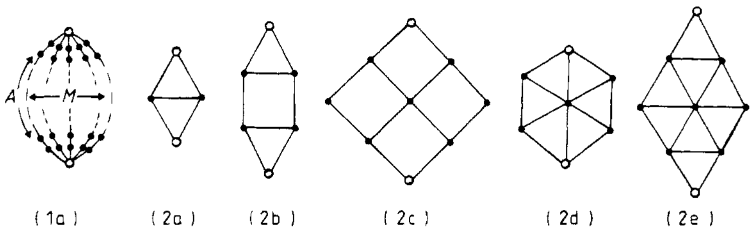

Figure 2. Some basic cells of hierarchies with $d_c = 1$ and 2. Examples are labelled with d_c and a letter. Nodes are shown as open circles (cf 2a with figure 1).

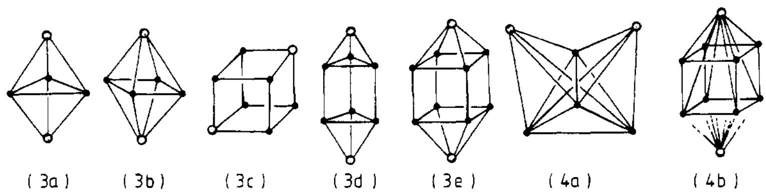

Figure 3. Some basic cells with $d_c = 3$ and 4.

Kaufman and Griffiths (1981) note the problem of defining length and dimension on the hierarchies; however, here such definitions will be made. The distance between two vertices is defined as the number of bonds on the shortest path on the lattice between the vertices. The scale change associated with the renormalisation step, b, is the distance between the nodes on the basic cell. The nth lattice in the iterative generation will be associated with a length b^n. This definition is independent of any embedding of the hierarchy in a Euclidean space. (Note that the number of minimum paths between the nodes of the nth lattice will grow, in the simple cases, as $p^{(b^n-1)/b-1)}$, where p is the number of minimum paths on the basic cell.)

Define respectively the intrinsic dimension, D, and connectivity, Q, by

$$D = \log(g)/\log(b) \qquad \text{and} \qquad Q = \log(q)/\log(b). \tag{1}$$

D and Q are defined on the cells and are the exponents which govern how respectively the number of bonds on the lattices and the minimum cut on the lattices grow as powers of b^n. (Note alternative definitions of D and Q (Mackenzie 1981) based on the growth of 'volume' and 'surface' with distance from an arbitrary vertex are problematic on the hierarchies due to their inhomogeneity.) On regular lattices $D(=d) = 1 + Q$; whilst this holds on the MKH in general one finds $D > 1 + Q$.

Table 1. Lattice parameters and Ising fixed points and eigenvalues.

Lattice	2a	2b	2c	2d	2e	3a	3b	3c	3d	3e	4a	4b
g	5	8	12	12	16	9	12	12	15	20	14	28
q	2	2	2	3	2	3	4	3	3	4	4	8
b	2	2	4	2	4	2	2	3	3	3	2	2
J^*	0.4407	0.6673	0.7111	0.3269	0.5241	0.2351	0.1832	0.4326	0.3781	0.3184	0.1606	0.0941
λ_t	1.8284	2.0736	2.3352	2.4038	2.4293	2.1577	2.2173	2.6096	2.8164	3.1917	2.3312	2.2764
λ_h	4.4853	7.6254	11.0557	9.1896	14.5783	6.6577	7.8005	9.5394	12.5988	15.0880	8.7398	12.9353

Table 2. D, Q and exponents.

Lattice	2c	2b	2e	3c	2a	3d	3e	3a	3b	2d	4a	4b
D	1.79	1.89	2	2.26	2.32	2.46	2.73	3.17	3.58	3.58	3.81	4.81
Q	0.5	0.63	0.5	1	1	1	1.26	1.58	2	1.58	2	3
J^*	0.71	0.67	0.52	0.43	0.44	0.38	0.32	0.23	0.18	0.33	0.16	0.09
Y_t	0.62	0.66	0.64	0.873	0.871	0.94	1.06	1.12	1.15	1.26	1.22	1.19
Y_h	1.73	1.85	1.93	2.03	2.16	2.31	2.47	2.73	2.96	3.20	3.13	3.69
α	−0.90	−0.85	−1.12	−0.59	−0.67	−0.61	−0.58	−0.86	−1.12	−0.83	−1.12	−2.05
β	0.09	0.06	0.10	0.24	0.18	0.17	0.24	0.39	0.54	0.30	0.56	0.94
δ	29.3	42.4	28.8	9.83	13.8	14.5	9.63	6.29	4.77	8.31	4.60	3.31
γ	2.71	2.72	2.91	2.11	2.31	2.28	2.10	2.07	2.04	2.22	2.005	2.17
D/Q	3.58	3	4	2.26	2.32	2.46	2.17	2.01	1.79	2.26	1.90	1.60
Y_h/Y_t	2.79	2.80	3.01	2.32	2.42	2.46	2.33	2.43	2.57	2.54	2.56	3.11

Both D and Q play roles in phase transitions on the hierarchies. Discussion below is made in the context of the Ising model. The treatment of external fields on the hierarchies is awkward due to the inhomogeneity of the coordinations. Following Yeomans and Fisher (1981) and implicitly Jose *et al* (1977) the Ising magnetic field is assigned to the spins in proportion to their coordinations. This allows the Ising Hamiltonian to be written

$$-\beta H = \sum_{ij} JS_iS_j + h(S_i + S_j) \qquad (S_i = \pm 1). \tag{2}$$

Unlike other field assignments (3) remains of the same form under the decimation transformation. It is straightforward to generate recursion relations on a computer and the conventional analysis was pursued by Niemeyer and van Leeuwen (1973). On each example a single unstable fixed fixed point, J^*, was found.

The author has obtained the following results.

(i) At the $T=0$ (discontinuity) fixed point Nienhuis and Nauenberg (1975) and Klein *et al* (1976) respectively argue that on regular lattices the scaling eigenvalues obey $\lambda_h = b^d$ and $\lambda_t = b^{d-1}$. On the hierarchies considered here the recursion relations for the Ising model have the general forms

$$J'(h=0) = \frac{1}{2}\ln\left(\frac{\exp(gJ) + \text{LO}}{r\exp[(g-2q)J] + \text{LO}}\right), \tag{3}$$

$$h' = \frac{1}{4}\ln\left(\frac{\exp(gJ + 2gh) + \text{LO}}{\exp(gJ - 2gh) + \text{LO}}\right), \tag{4}$$

where r is the number of minimum cuts on the basic cell and LO means lower order in $\exp(J)$. From (3) and (4) one finds $\lambda_h = g$ and $\lambda_t = q$, or from (1), $\lambda_h = b^D$ and $\lambda_t = b^Q$ generalising the regular lattice results.

(ii) Finitely ramified hierarchies, for the examples here, have $q=1$. Such examples have $T_c = 0$ and essential singularities (Gefen *et al* 1980). A change of variable, $\tilde{t} = \exp(-2J)$, is used to describe these singularities. From (3) one finds $\lambda_{\tilde{t}} = r$. This makes specific for all branching Koch curves ($q=1$ here) the variations reported by Gefen *et al* (1980) (see also Gefen *et al* 1983).

Kaufman and Griffiths (1981) and Dhar (1977) both note that due to the lack of translational invariance the usual correlation function cannot be defined on the hierarchies and hence the length exponents Y_t and Y_h do not have immediate interpretations. The thermodynamic exponents, however, are well defined (simply substitute λ_t and λ_h with g in the usual relations), do have their usual interpretations and are independent of b and D. It is therefore of interest to see how the thermodynamic exponents vary with D and Q.

(iii) Table 2 presents exponents found of the cells of figures 1 and 2. The table is ordered on increasing D which, as is evident, gives a rough ordering to the exponents. The overall variations with increasing D (increasing β, Y_h and Y_t, decreasing J^*, δ and γ and a maxima in α) are those expected from consideration of results on regular lattices and field theories (Domb 1973, Wilson and Kogut 1974). Values for examples with D around 4 do not correspond to those of the ε expansion (which is based on a continuium treatment of regular lattices). Clearly exponents vary with both D and Q (and it is anticipated other parameters which remain to be found). In particular the droplet model of Fisher (1967) suggests the relationship $d/(d-1) \simeq Y_h/Y_t$) on regular lattices; this generalises to the hierarchies as $D/Q \simeq Y_h/Y_t$. As seen from the table this holds weakly (note that as on regular lattices as D increases Y_h/Y_t exceeds what is within the droplet model its physical maxima). However, given this relationship and that $D > 1+Q$ it could be anticipated that for a hierarchy of a given D the value of Y_h would be shifted up and the value of Y_t shifted down with respect to those values interpolated for the same D on regular lattices with $D = 1+Q$. Such a trend is clearly shown in the table (e.g. 2e for which $Y_t < 1$ yet $Y_h > 1.875$). Furthermore, a breakdown in ordering shown by β and δ of 3c, 2a and 3d can be understood as for these D/Q increases with D in contrast to the trend on regular lattices. Similarly 2d appears misplaced but has D/Q relatively large.

It is interesting to investigate families of hierarchies which contain in the limit of large basic cells members with specific limits of D and Q. On these examples Y_t and Y_h will be discussed, other exponents being found as usual from these, D and the scaling relations. Details of calculations will be published elsewhere.

(iv) A family, the ladders, on which $D \to 1$ and $Q \to 0$ is shown in figure 4(a). Calculations on these, using transfer matrix multiplication to find recursion relations, reveal that as $D \to 1$ $J^* \to \infty$ $\lambda_h \to g$ and $\lambda_t \to q(=2)$ or $Y_h \to D$ and $Y_t \to Q$. Though infinitely ramified the family seems to be approaching a lower critical dimension defined by $Q = 0$, suggesting that this condition rather than finite ramification (Gefen *et al* 1980) should serve to generalise the concept of lower critical dimension to hierarchies. Furthermore the above exponents generalise the $1+\varepsilon$ results found on regular lattices (Migdal 1975, Wallace and Zia 1979).

(v) The MKH form a family completely parametrised by two parameters (M and A in 1a of figure 2). Melrose (1983a) finds that contours of constant exponents in the (M, A) space are unique for each exponent and do not follow the contours of constant D and Q, although they are close to these contours for low M and A.

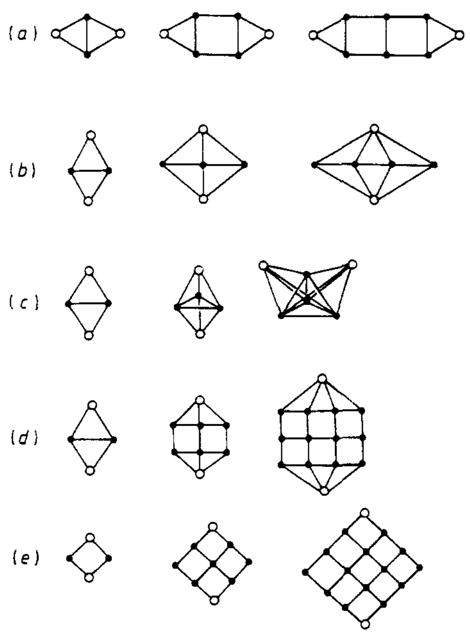

Figure 4. Some families of hierarchies: (a) ladders, (b) simple strings, (c) hyperpyramids, (d) self-duals, (e) squares.

(vi) The multiplicity set by M on the MKH can define a family for any basic cell by simply considering M cells in parallel connected at the nodes. M enters recursion relations simply as a multiplicative constant. As $M \to \infty$, D and $Q \to \infty$ with $D/Q \to 1$. On all examples studied the author finds that as $M \to \infty J^* \to 0$, $\lambda_h \to Mq$ and $\lambda_t \to b$ or $Y_h \to Q$ and $Y_t \to 1$ (a result well known on the MKH (Migdal 1975)).

(vii) Two other families with $D \to \infty$ are shown in figures 4(b) and 4(c). The first, the simple strings, has $D \to \infty$ with $D/Q \to 1$. On these the author finds the same limits as under increasing M above. (Note that cells with $d_c = 2$ show a duality akin to that of planar lattices. The simple strings are dual to the ladders. It seen that in general such dual hierarchies do not have the same D and Q, Melrose (1983b). The second family, the hyperpyramids, has $D \to \infty$ with $D/Q \to 2$. Explict recursion relations may be found for this family, although computation at high d_c is still difficult. The author finds that $\lambda_h \to 2q$ and that $Y_h \to Q+1$ (or $Y_h \to D/2+1$) this gaussian result being consistent with the growing coordinations). Extrapolation suggests, however, that $1.58 < Y_t(D \to \infty) < 1.64$. One may find a variety of high D limits, in particular examples with Q finite as $D \to \infty$.

(viii) Two families with $D \to 2$ are shown in figures 4(d) and 4(e). The first, the self-duals, has been well studied in the guise of a square lattice approximation (Martin and Tsallis 1981 and references therein). As $D \to 2$, $Q \to 1$, these limits add some justification to the suggested convergence of exponents on these cells to those of two-dimensional regular lattices. Table 3 shows Y_t and Y_h for the first three cells; note that Y_h follows the initial peak in D(Y_h here differs from that of Martin and Tsallis (1981) due to a different field assignment). The second family, the squares, shows a different behaviour on these as $D \to 2$, $Q \to 0$. Table 4 shows Y_t and Y_h again for the first three cells. Note that Y_h follows the initial dip in D. It is interesting to note that although in the large cell limit the cell is a square lattice, the hierarchy having $Q = 0$ will show, presumably, the behaviour of a lower critical dimension.

In conclusion, the breakdown in universality suggested by Gefen *et al* (1980) has here been exhibited on exactly solvable hierarchies with $T_c > 0$. The definitions of b, D and Q have been found useful in discussing exponent variations under the freedom in Q, generalising regular lattice results at discontinuity fixed points and in $1+\varepsilon$ dimensions, and classifying large cell limits of families. The hierarchies clearly can

Table 3.

D	Q	J^*	Y_t	Y_h
2.3291	1	0.4407	0.8706	2.1652
2.3317	1	0.4407	0.9042	2.1847
2.3073	1	0.4407	0.9132	2.1769

Table 4.

D	Q	J^*	Y_t	Y_h
2	1	0.6094	0.7472	1.8791
1.7925	0.5	0.7111	0.6179	1.7333
1.7737	0.3868	0.7110	0.5595	1.7277

show a variety of high D limits. Of course, D and Q are not complete enough to determine all exponent variations (indeed, the droplet model of Fisher (1967) and particularly the result (v) above indicate this). Further parameters need to be sought. A parameter defined by Dhar (1977) and, separately, Alexander (1982) will be discussed in a future publication.

Acknowledgment

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Note added in proof. (I) For the cells of figures 2 and 3 the cell dimension can be defined as the dimension of the least Euclidian space in which the cell may be embedded such that all bonds are of the same length. However in general such a useful definition is problematical. (II) The results and parameters given here allow an understanding of exponent variations shown by the cells as finite cluster approximations on regular lattices.

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